ALGEBRA
BERNAYS
SVEUČILIŠTE

MATEMATIČKA ANALIZA

Površina lika omeđenog krivuljama

Newton – Leibnizova formula

Neka je funkcija f definirana i neprekidna na $[a, b]$.
Tada postoji primitivna funkcija F funkcije f i vrijedi:

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Metode integriranja

- Neposredna integracija
- Metoda supstitucije
  - zamjena granica integrala
- Parcijalna integracija

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

Zadatak 1. $\int_0^2 x^3\sqrt{x^4+9}\,dx$

$$= \left| \begin{array}{c} x^4 + 9 = t \\ 4x^3 dx = dt \\ dx = \dfrac{dt}{4x^3} \\ x = 0 \;\rightarrow\; t = 0^4 + 9 = 9 \\ x = 2 \;\rightarrow\; t = 2^4 + 9 = 25 \end{array} \right| = \int_9^{25} x^3\sqrt{t}\,\frac{dt}{4x^3}$$

$$= \frac{1}{4}\frac{t^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_9^{25} = \frac{1}{6}\sqrt{t^3}\Bigg|_9^{25} = \frac{125}{6} - \frac{27}{6} = \frac{49}{3}$$

Zadatak 2. $\int_0^2 x^3\sqrt{x^4+9}\,dx = (*)$

$$\int x^3\sqrt{x^4+9}\,dx = \left|\begin{matrix} x^4+9=t \\ 4x^3dx=dt \\ dx=\dfrac{dt}{4x^3} \end{matrix}\right| = \int x^3\sqrt{t}\frac{dt}{4x^3}$$

$$= \frac{1}{4}\int \sqrt{t}dt = \frac{1}{4}\cdot\frac{t^{\frac{3}{2}}}{\frac{3}{2}} + c = \frac{1}{6}\sqrt{t^3} + c = \frac{1}{6}\sqrt{(x^4+9)^3} + c$$

$$(*) = \frac{1}{6}\sqrt{(x^4+9)^3}\,\Big|_0^2 = \frac{1}{6}\sqrt{25^3} - \frac{1}{6}\sqrt{9^3} = \frac{49}{3}$$

Zadatak 3.

$$\int_1^e \frac{\ln x}{x^3}dx = \begin{vmatrix} u = \ln x & dv = x^{-3}dx \\ du = \frac{1}{x}dx & v = \frac{x^{-2}}{-2} = -\frac{1}{2x^2} \end{vmatrix}$$

$$= \left(-\frac{1}{2x^2}\ln x\right)\Big|_1^e - \int_1^e -\frac{1}{2x^2}\cdot\frac{1}{x}dx$$

$$= \left(-\frac{1}{2x^2}\ln x\right)\Big|_1^e + \frac{1}{2}\int_1^e x^{-3}dx$$

Zadatak 3.

$$= \left(-\frac{1}{2x^2}\ln x\right)\Big|_1^e + \frac{1}{2}\int_1^e x^{-3}dx$$

$$= \left(-\frac{\ln x}{2x^2}\right)\Big|_1^e + \frac{1}{2}\cdot\frac{x^{-2}}{-2}\Big|_1^e = \left(-\frac{\ln x}{2x^2}\right)\Bigg|_1^e - \frac{1}{4x^2}\Bigg|_1^e$$

$$= \left(-\frac{\ln e}{2e^2}\right) - \left(-\frac{\ln 1}{2\cdot 1^2}\right) - \left(\frac{1}{4e^2} - \frac{1}{4\cdot 1^2}\right)$$

$$= -\frac{1}{2e^2} - \frac{1}{4e^2} + \frac{1}{4} = -\frac{3}{4e^2} + \frac{1}{4}$$

Zadatak 4. $\int_{\frac{\pi}{2}}^{\pi} (x-3)\cos 2x \, dx$

$$\int (x-3)\cos 2x \, dx = \left| \begin{matrix} u = x-3 & dv = \cos 2x \, dx \\ du = dx & v = \frac{1}{2}\sin 2x \end{matrix} \right|$$

$$= \frac{x-3}{2}\sin 2x - \frac{1}{2}\int \sin 2x \, dx$$

$$= \frac{x-3}{2}\sin 2x + \frac{1}{4}\cos 2x + c$$

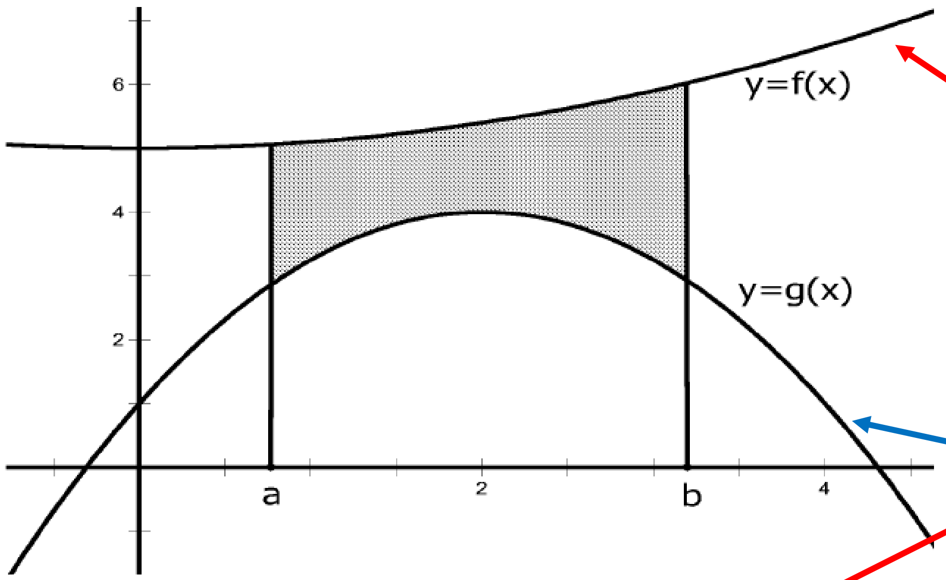
Zadatak 4.

$$\int_{\frac{\pi}{2}}^{\pi} (x-3)\cos 2x \, dx = \left(\frac{x-3}{2}\sin 2x + \frac{1}{4}\cos 2x\right)\Bigg|_{\frac{\pi}{2}}^{\pi}$$

$$= \left(\frac{\pi-3}{2}\sin 2\pi + \frac{1}{4}\cos 2\pi\right) - \left(\frac{\frac{\pi}{2}-3}{2}\sin \pi + \frac{1}{4}\cos \pi\right)$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

Površina lika omeđenog krivuljama

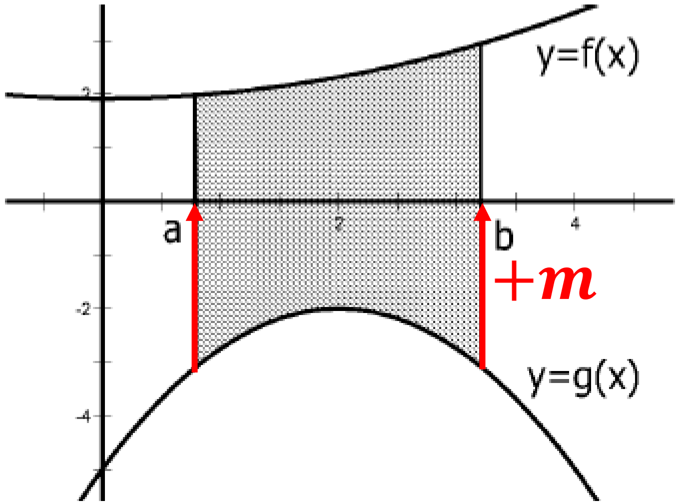


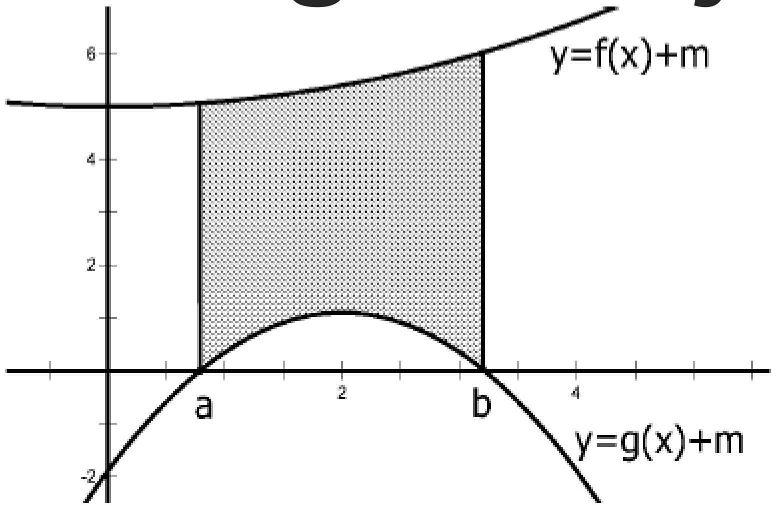
gornja krivulja

donja krivulja

$$P = \int_a^b (f(x) - g(x))dx$$

Površina lika omeđenog krivuljama





$$P = \int_a^b \big((f(x) + m) - (g(x) + m)\big)dx$$

$$P = \int_a^b (f(x) - g(x))dx$$

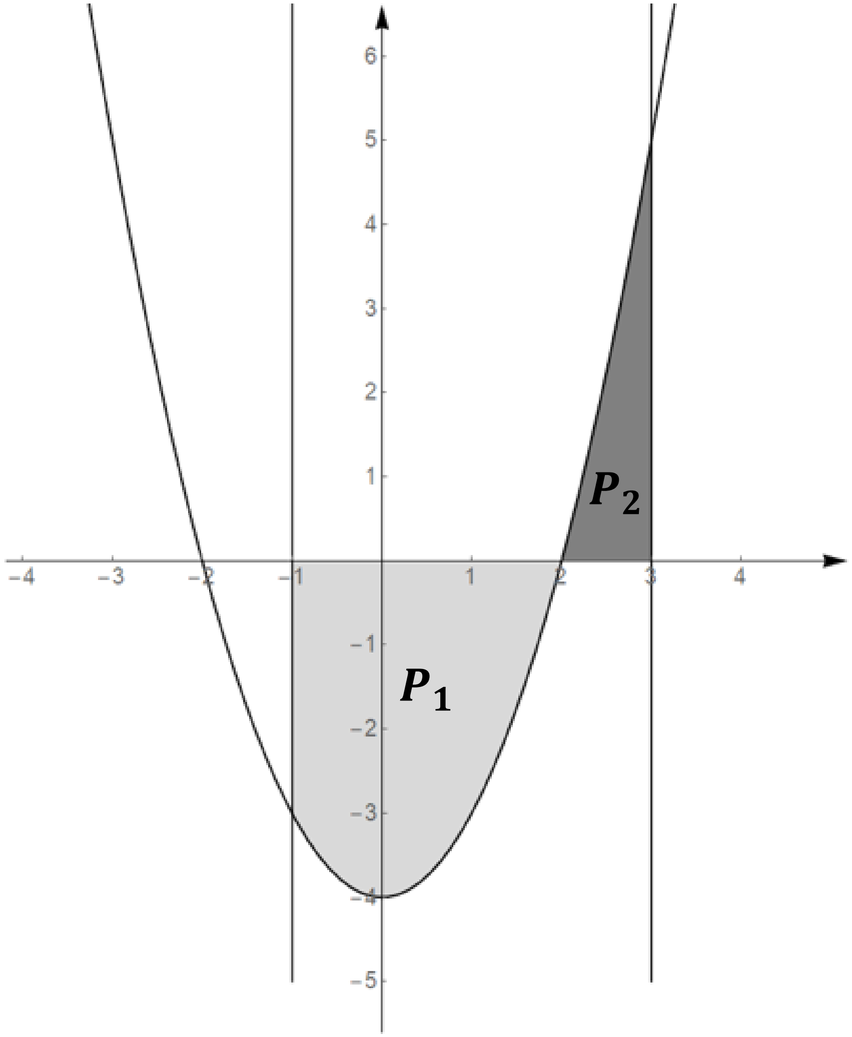
Zadatak 5. Izračunajte površinu lika omeđenog krivuljama:

$y = x^2 - 4,\ x = -1,\ x = 3,\ y = 0.$

$$P_1 = \int_{-1}^{2} \left(0 - (x^2 - 4)\right) dx$$

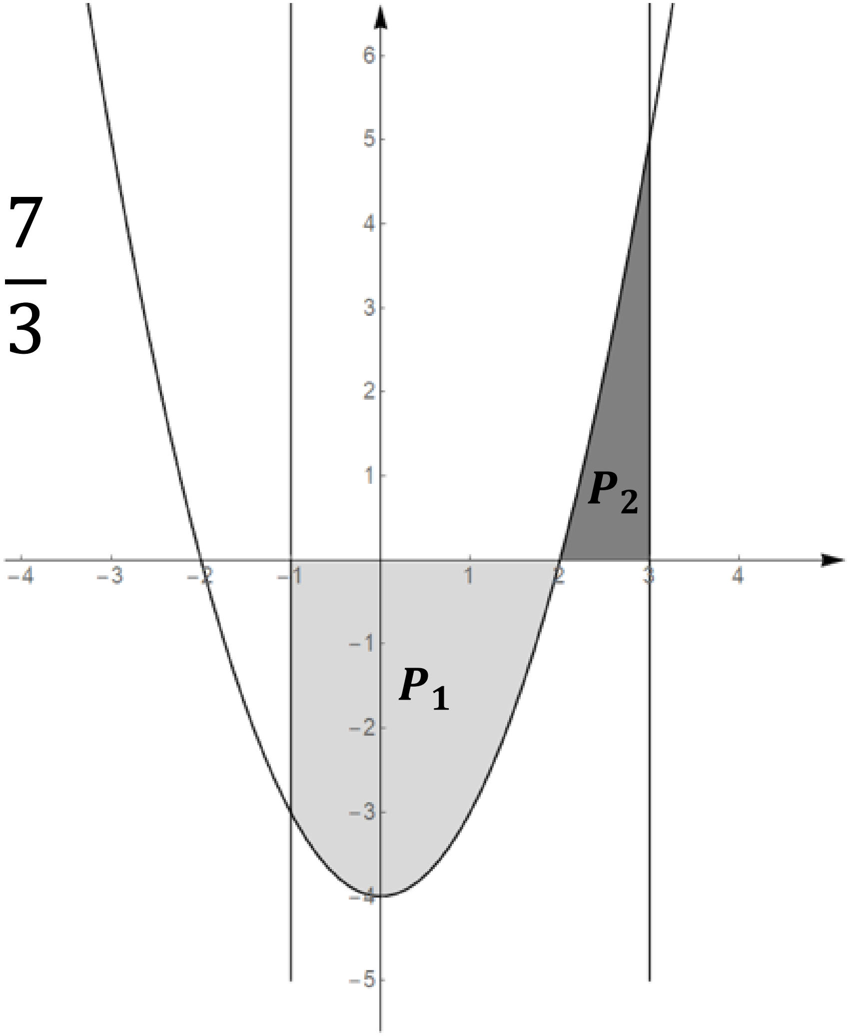
$$= \left(-\frac{x^3}{3} + 4x\right)\Bigg|_{-1}^{2}$$

$$= 9$$



Zadatak 5. Izračunajte površinu lika omeđenog krivuljama:

$y = x^2 - 4,\ x = -1,\ x = 3,\ y = 0.$

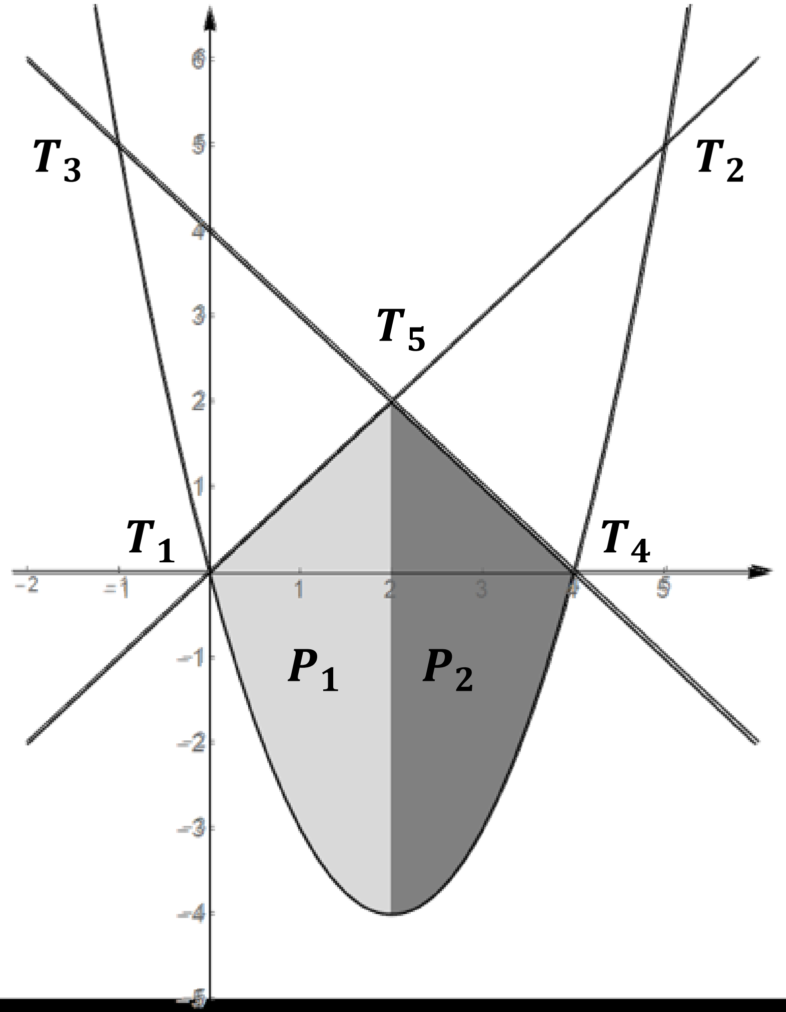


$$P_2 = \int_2^3 (x^2 - 4 - 0)dx = \left(\frac{x^3}{3} - 4x\right)\Bigg|_2^3 = \frac{7}{3}$$

$$P_1 + P_2 = \frac{34}{3}$$

$$\int_{-1}^{3} (x^2 - 4)dx = -\frac{20}{3}$$

Zadatak 5. Izračunajte površinu najvećeg lika omeđenog krivuljama: $y = x,\ \ y = x^2 - 4x,\ y = 4 - x.$



Presjecišta: $y = x,\ y = x^2 - 4x$

$$x^2 - 4x = x$$

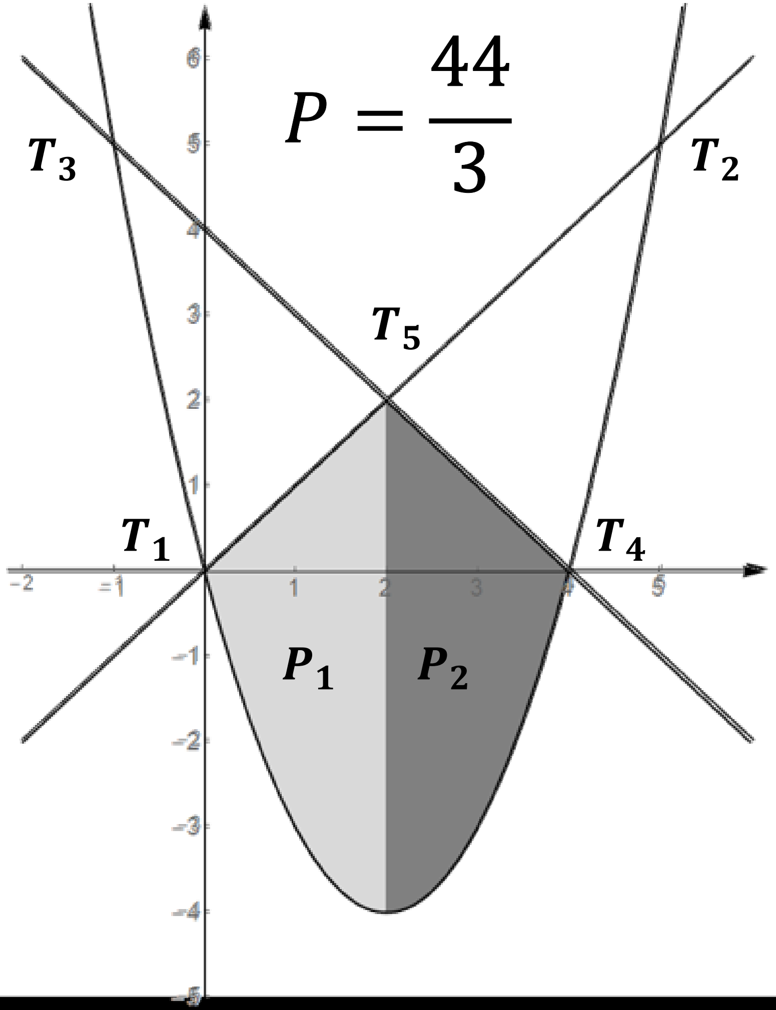
$x_1 = 0$ $\quad$ $x_2 = 5$

$y_1 = 0$ $\quad$ $y_2 = 5$

$T_1(0{,}0)$ $\quad$ $T_2(5{,}5)$

$T_3(-1{,}5)$ $\quad$ $T_4(4{,}0)$ $\quad$ $T_5(2{,}2)$

Zadatak 5. Izračunajte površinu najvećeg lika omeđenog krivuljama: $y = x,\ \ y = x^2 - 4x,\ \ y = 4 - x.$



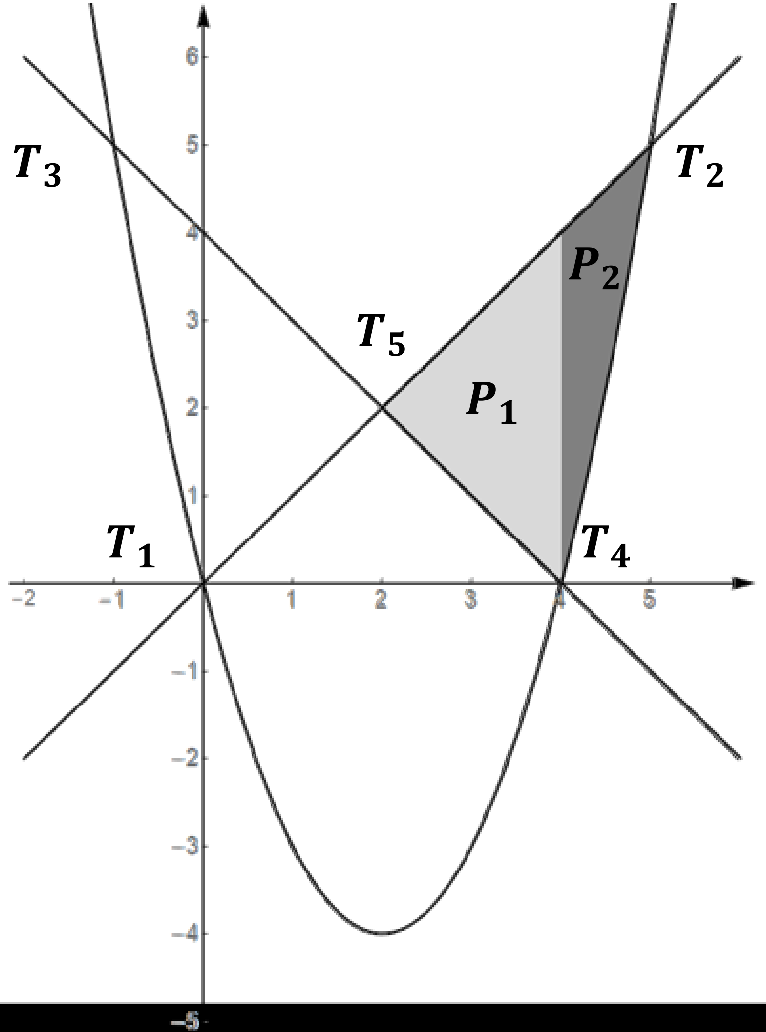
$$P_1 = \int_0^2 \left(x - (x^2 - 4x)\right)dx$$

$$= \left(5\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^2 = \frac{22}{3}$$

$$P_2 = \int_2^4 \left((4 - x) - (x^2 - 4x)\right)dx$$

$$= \left(4x + 3\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_2^4 = \frac{22}{3}$$

Zadatak 5. Izračunajte površinu lika u prvom kvadrantu omeđenog krivuljama: $y = x, \; y = x^2 - 4x, \; y = 4 - x.$



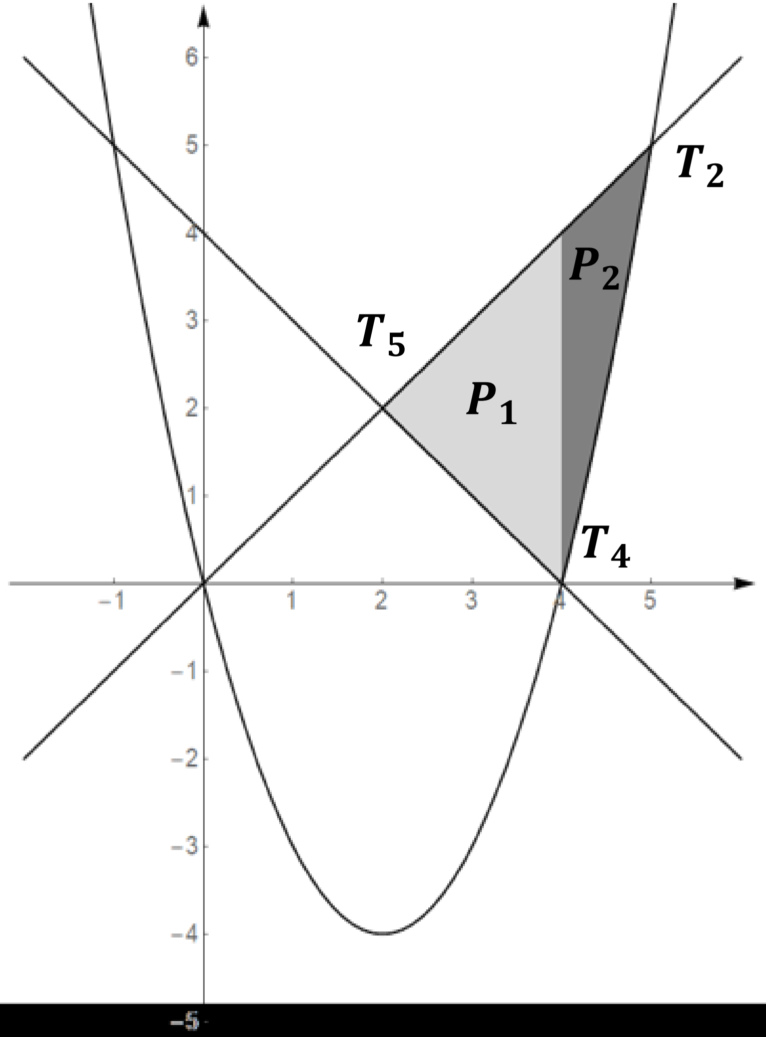
Presjecišta:

$T_2(5,5)$

$T_4(4,0)$

$T_5(2,2)$

Zadatak 5. Izračunajte površinu lika u prvom kvadrantu omeđenog krivuljama: $y = x,\ \ y = x^2 - 4x,\ y = 4 - x.$

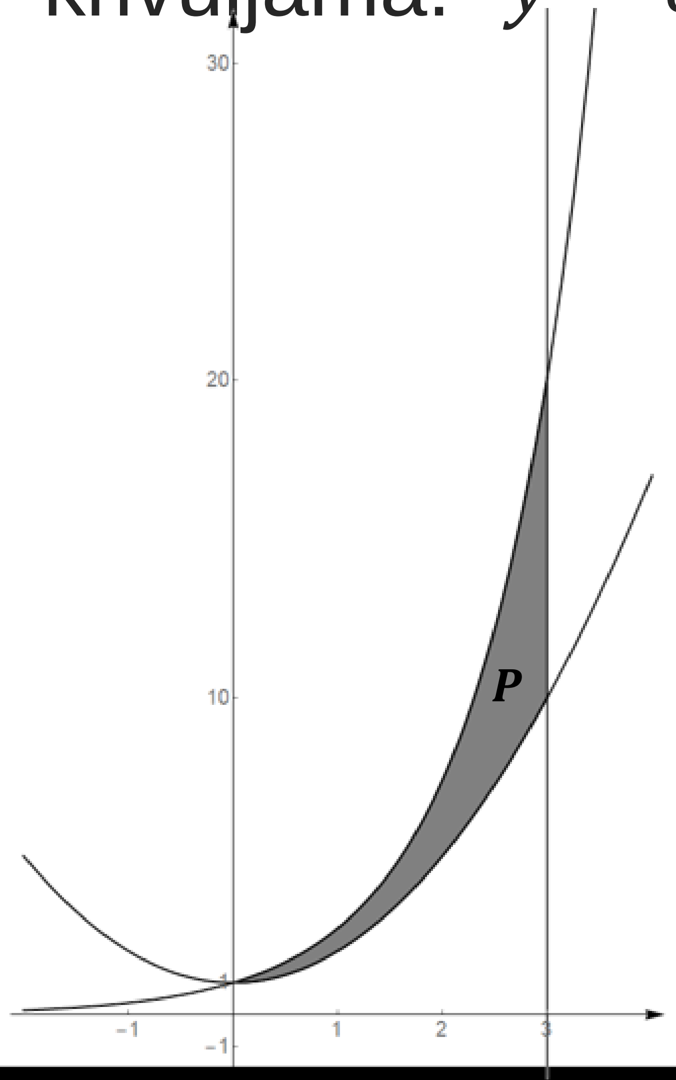


$$P_1 = \int_2^4 \big(x - (4 - x)\big)dx = 4$$

$$P_2 = \int_4^5 \big(x - (x^2 - 4x)\big)dx = \frac{13}{6}$$

$$P = P_1 + P_2 = \frac{37}{6}$$

Zadatak 6. Izračunajte površinu lika omeđenog krivuljama: $y = e^x,\ y = x^2 + 1,\ x = 3.$



$$P = \int_0^3 \left(e^x - (x^2 + 1)\right)dx$$

$$= \left(e^x - \frac{x^3}{3} - x\right)\Big|_0^3$$

$$= \left(e^3 - \frac{3^3}{3} - 3\right) - \left(e^0 - \frac{0^3}{3} - 0\right) \ = e^3 - 13$$

Video materijali

- https://www.youtube.com/watch?v=S6GpTQiktuY&list=PLcWN1hq0ODxlDuvgKaGyXS8sVHmxJi34l&index=4
- https://www.youtube.com/watch?v=Ivg5mhPcNWo&list=PLcWN1hq0ODxlDuvgKaGyXS8sVHmxJi34l&index=5
- https://www.youtube.com/watch?v=kT_2R_RcgUs&list=PLcWN1hq0ODxlDuvgKaGyXS8sVHmxJi34l&index=6
- https://www.youtube.com/watch?v=HRLbTuSQ2LA&list=PLcWN1hq0ODxlDuvgKaGyXS8sVHmxJi34l&index=7

Hvala ☺