Zadatak 1 Neka je X neprekidna slučajna varijabla s funkcijom gustoće $f(x) = \begin{cases} c \cdot x^3(1-x), & 0 \le x \le 1 \\ 0, & \text{inače} \end{cases}$. Odredite:

a) konstantu c

b) $P(X \le 0.5)$ i $P(X > 0.1)$ i $P(0.1 < X \le 0.5)$

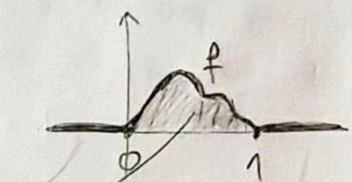
c) funkciju razdiobe (distribucije)

d) očekivanje

e) disperziju.

Rješenje:

a) "skica"



površina = 1

$$\boxed{\int_{-\infty}^{\infty} f(x) \cdot dx = 1}$$

$$0 + \int_0^1 f(x) \cdot dx + 0 = 1$$

$$\int_0^1 c \cdot x^3 (1-x) \cdot dx = 1$$

$$c \cdot \int_0^1 (x^3 - x^4) \cdot dx = 1$$

$$c \cdot \left(\frac{x^4}{4} - \frac{x^5}{5} \right) \Big|_0^1 = 1$$

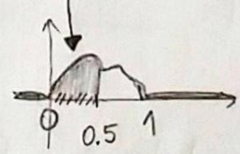
$$c \cdot \left[\frac{1^4}{4} - \frac{1^5}{5} - \left(\frac{0^4}{4} - \frac{0^5}{5} \right) \right] = 1$$

$$\frac{1}{20} \cdot c = 1 \quad / \cdot 20$$

$$\underline{c = 20}$$

$$f(x) = \begin{cases} 20x^3(1-x), & 0 \le x \le 1 \\ 0, & \text{inače} \end{cases}$$

b) $P(X \le 0.5) = \int_{-\infty}^{0.5} f(x) \cdot dx =$



$$= 0 + \int_0^{0.5} f(x) dx$$

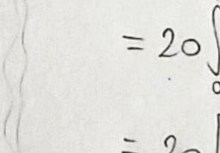
$$= \int_0^{0.5} 20x^3(1-x) dx$$

$$= 20 \int_0^{0.5} (x^3 - x^4) dx = 20 \left(\frac{x^4}{4} - \frac{x^5}{5} \right) \Big|_0^{0.5} =$$

$$= 20 \left[\frac{0.5^4}{4} - \frac{0.5^5}{5} - \left(\frac{0^4}{4} - \frac{0^5}{5} \right) \right]$$

$$= 20 \cdot \left(\frac{0.0625}{4} - \frac{0.03125}{5} \right) = 0.1875$$

$$P(X > 0.1) = \int_{0.1}^{\infty} f(x) dx = \int_{0.1}^{1} f(x) \cdot dx + 0 =$$



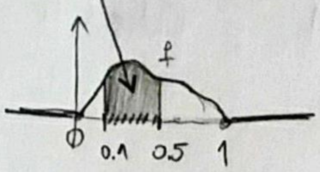
$$= \int_{0.1}^{1} 20x^3(1-x) dx = 20 \int_{0.1}^{1} (x^3 - x^4) dx$$

$$= 20 \left(\frac{x^4}{4} - \frac{x^5}{5} \right) \Big|_{0.1}^{1} = 20 \left[\frac{1^4}{4} - \frac{1^5}{5} - \left(\frac{0.1^4}{4} - \frac{0.1^5}{5} \right) \right]$$

$$= 20 \left(\frac{1}{4} - \frac{1}{5} - \frac{1}{40000} + \frac{1}{500000} \right)$$

$$= 0.99954$$

$$P(0.1 < X \leq 0.5) = \int_{0.1}^{0.5} f(x)\cdot dx = \int_{0.1}^{0.5} 20x^3(1-x)\,dx = 20\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Bigg|_{0.1}^{0.5} =$$



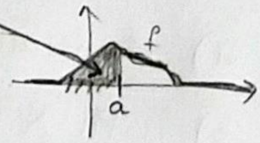
$$= 20\left[\frac{0.5^4}{4} - \frac{0.5^5}{5} - \left(\frac{0.1^4}{4} - \frac{0.1^5}{5}\right)\right] = \ldots = 0.18704$$

općenito:

$F(a) = P(X \leq a)$

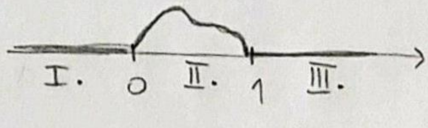
površina (vjerojatnost) lijevo od a

tj. $\boxed{F(a) = \int_{-\infty}^{a} f(x)\cdot dx}$

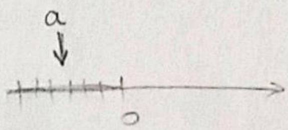


c) funkcija razdiobe $F(a) = ?$

$$f(x) = \begin{cases} 20x^3(1-x), & 0 \leq x \leq 1 \\ 0, & \text{inače} \end{cases}$$



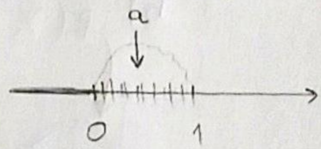
$(a \leq 0)$ I. $a \in \langle -\infty, 0]$



$$\boxed{F(a) = \underbrace{\int_{-\infty}^{a} f(x)\cdot dx}_{0}} = \underline{\underline{0}}$$

(površina lijevo od a, za bilo koji broj a od $-\infty$ do 0 je jednaka $\underline{\underline{0}}$)

$(0 < a \leq 1)$ II. $a \in \langle 0, 1]$



$$F(a) = \int_{-\infty}^{a} f(x)\cdot dx = \underbrace{\int_{-\infty}^{0} \underbrace{f(x)}_{0}\,dx}_{0} + \int_{0}^{a} f(x)\cdot dx =$$

$$= 0 + \int_{0}^{a} 20x^3(1-x)\cdot dx$$

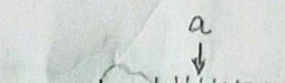
$$= 20\int_{0}^{a}(x^3 - x^4)\,dx = 20\left(\frac{x^4}{4} - \frac{x^5}{5}\right)\Bigg|_{0}^{a}$$

$$= 20\left[\frac{a^4}{4} - \frac{a^5}{5} - \left(\frac{0^4}{4} - \frac{0^5}{5}\right)\right] = 20\left(\frac{a^4}{4} - \frac{a^5}{5}\right) = \cancel{20}^{5}\frac{a^4}{\cancel{4}} - \cancel{20}^{4}\frac{a^5}{\cancel{5}}$$

$$F(a) = 5a^4 - 4a^5$$

[površina lijevo od a, kada je a bilo koji broj između 0 i 1: obuhvaćen je cijeli interval od $-\infty$ do 0 te od 0 pa do a (tj. od 0 pa do bilo kojeg broja između 0 i 1)]

$(a > 1)$ III. $a \in \langle 1, \infty \rangle$



$$F(a) = \int_{-\infty}^{a} f(x)\cdot dx$$

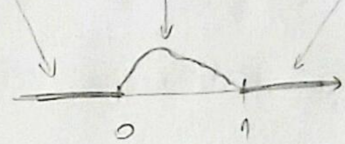
$$= \underbrace{\int_{-\infty}^{0} \underbrace{f(x)}_{0}\cdot dx}_{0} + \underbrace{\int_{0}^{1} f(x)\cdot dx}_{\text{sva površina} = 1} + \underbrace{\int_{1}^{a} \underbrace{f(x)}_{0}\cdot dx}_{0} = 1$$

[površina lijevo od a, kada je a bilo koji broj od 1 do ∞: obuhvaćen je cijeli interval od $-\infty$ do 1, pa cijeli od 0 do 1 (što zahvaća cijelu površinu koja je uvijek 1) te interval od 1 pa do a (tj. od 1 pa do bilo kojeg broja između 1 i ∞)

funkcija distribucije:

$$F(a)=\begin{cases} 0 & , \quad a\le 0 \\ 5a^4-4a^5 & , \quad 0<a\le 1 \\ 1 & , \quad a>1 \end{cases}$$

d) očekivanje: $\boxed{E(X)=\int_{-\infty}^{\infty} x\cdot f(x)\cdot dx} = 0 + \int_0^1 x\cdot f(x)dx + 0 =$

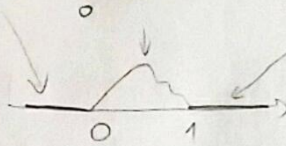


$$= \int_0^1 x\cdot 20x^3(1-x)dx = 20\int_0^1 x^4(1-x)dx$$

$$= 20\int_0^1 (x^4-x^5)dx = 20\left(\frac{x^5}{5}-\frac{x^6}{6}\right)\Big|_0^1 = 20\left[\frac{1^5}{5}-\frac{1^6}{6}-\left(\frac{0^5}{5}-\frac{0^6}{6}\right)\right]$$

$$= 20\left(\frac{1}{5}-\frac{1}{6}\right) = 20\cdot\frac{1}{30} = \frac{2}{3}$$

e) $\boxed{E(X^2)=\int_{-\infty}^{\infty} x^2\cdot f(x)\cdot dx} = 0 + \int_0^1 x^2\cdot f(x)\cdot dx + 0 = \int_0^1 x^2\cdot 20x^3(1-x)dx =$



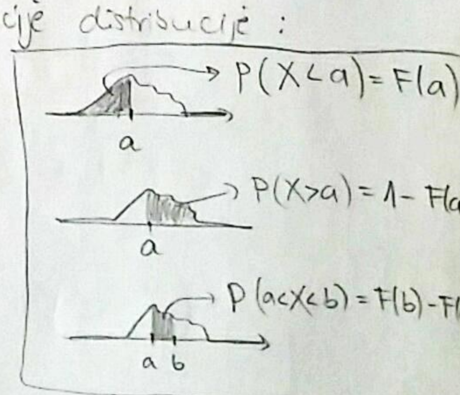
$$= 20\cdot\int_0^1 x^5(1-x)dx = 20\int_0^1 (x^5-x^6)dx = 20\left(\frac{x^6}{6}-\frac{x^7}{7}\right)\Big|_0^1 = 20\cdot\left[\frac{1^6}{6}-\frac{1^7}{7}-\left(\frac{0^6}{6}-\frac{0^7}{7}\right)\right]$$

$$= 20\cdot\left(\frac{1}{6}-\frac{1}{7}\right) = 20\cdot\frac{1}{42} = \frac{10}{21}$$

disperzija: $\boxed{D(X)=E(X^2)-[E(X)]^2} = \frac{10}{21}-\left(\frac{2}{3}\right)^2 = \frac{10}{21}-\frac{4}{9} = \frac{2}{63} \approx 0.032$

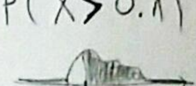
---

komentar: zadatak b) se mogao riješiti pomoću funkcije distribucije:
$F(a)=5a^4-4a^5$, za a između 0 i 1

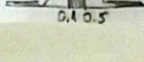


$P(X\le 0.5) = F(0.5) = 5\cdot 0.5^4-4\cdot 0.5^5 = 0.1875$

$P(X>0.1) = 1-F(0.1) = 1-(5\cdot 0.1^4-4\cdot 0.1^5) = 0.99954$



$P(0.1<X<0.5) = F(0.5)-F(0.1) = 5\cdot 0.5^4-4\cdot 0.5^5-(5\cdot 0.1^4-4\cdot 0.1^5)$
$= 0.18704$



Zadatak 2 | Neka je X neprekidna slučajna varijabla s funkcijom gustoće $f(x) = \begin{cases} c \cdot x, & 0 \le x \le 1 \\ 2 - cx, & 1 < x \le 2 \\ 0, & \text{inače} \end{cases}$. Odredite:

a) konstantu c

b) $P(-3 \le X < \frac{1}{3})$ i $P(X < 1.5)$ i $P(X > -2)$

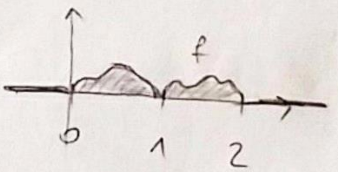
c) funkciju distribucije (razdiobe)

d) očekivanje

e) disperziju.

Rješenje:

a) "skica"



ukupna površina $= 1$

$$\boxed{\int_{-\infty}^{\infty} f(x) \cdot dx = 1}$$

$$0 + \int_0^1 f(x) \cdot dx + \int_1^2 f(x) dx + 0 = 1$$

$$\int_0^1 c \cdot x \cdot dx + \underbrace{\int_1^2 (2 - cx) dx}_{\int_1^2 2 \cdot dx - \int_1^2 c \cdot x \cdot dx} = 1$$

$$c \int_0^1 x \cdot dx + 2\int_1^2 dx - c \cdot \int_1^2 x dx = 1$$

$$c \cdot \frac{x^2}{2}\Big|_0^1 + 2x\Big|_1^2 - c \cdot \frac{x^2}{2}\Big|_1^2 = 1$$

$$c \cdot \left(\frac{1^2}{2} - \frac{0^2}{2}\right) + 2(2-1) - c\left(\frac{2^2}{2} - \frac{1^2}{2}\right) = 1$$

$$\frac{1}{2}c + 2 - c \cdot \left(2 - \frac{1}{2}\right) = 1$$

$$\frac{1}{2}c + 2 - \frac{3}{2}c = 1 \quad / \cdot 2$$

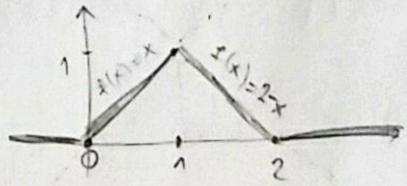
$$c + 4 - 3c = 2$$

$$-2c = -2 \quad / : (-2)$$

$$\underline{c = 1}$$

$$f(x)=\begin{cases} x \quad (\text{pravci}), & 0\le x\le 1 \\ 2-x, & 1<x\le 2 \\ 0, & \text{inače}\end{cases}$$

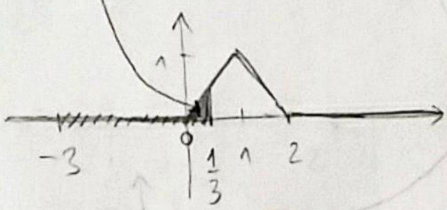
prava skica bi bila:



(5)

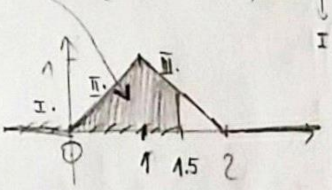
b)

$$P\left(-3\le X<\frac{1}{3}\right)=0+\int_0^{\frac{1}{3}} f(x)\cdot dx=\int_0^{\frac{1}{3}} x\cdot dx=\frac{x^2}{2}\Big|_0^{\frac{1}{3}}=\frac{\left(\frac{1}{3}\right)^2}{2}-\frac{0^2}{2}=$$



$$=\frac{\frac{1}{9}}{2}=\frac{1}{18}$$

$$P(X<1.5)=\underset{\mathrm{I.}}{0}+\underbrace{\int_0^1 f(x)dx}_{\mathrm{II.}}+\underbrace{\int_1^{1.5} f(x)dx}_{\mathrm{III.}}=\int_0^1 x\cdot dx+\int_1^{1.5}(2-x)\,dx=\frac{x^2}{2}\Big|_0^1+\left(2x-\frac{x^2}{2}\right)\Big|_1^{1.5}$$



$$=\frac{1^2}{2}-\frac{0^2}{2}+2\cdot 1.5-\frac{1.5^2}{2}-\left(2\cdot 1-\frac{1^2}{2}\right)=\frac{1}{2}+3-1.125-2+\frac{1}{2}=0.875$$

ili brži način: komplement

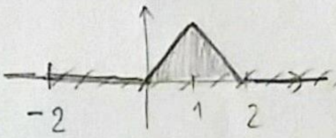
$$P(X<1.5)=\underbrace{1-\int_{1.5}^{2} f(x)\,dx}_{1-P(X\ge 1.5)}=1-\int_{1.5}^{2}(2-x)dx=1-\left(2x-\frac{x^2}{2}\right)\Big|_{1.5}^{2}=$$

$$=1-\left[2\cdot 2-\frac{2^2}{2}-\left(2\cdot 1.5-\frac{1.5^2}{2}\right)\right]$$

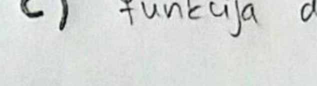
$$=1-(4-2-3+1.125)=0.875$$

$$P(X>-2)=1$$

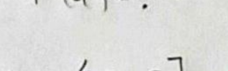
jer je obuhvaćena sva površina



c) funkcija distribucije $F(a)=?$

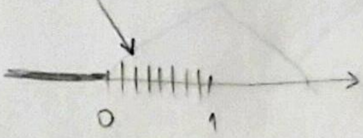


I. $a\in\langle-\infty,0]$ (tj. $a\le 0$)



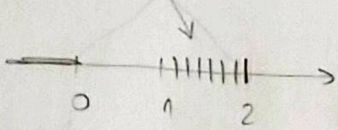
$$\boxed{F(a)=\int_{-\infty}^{a}\underbrace{f(x)}_{0}\cdot dx}=0$$

II. $a \in \langle 0,1]$ (tj. $0 < a \le 1$)



$$\boxed{F(a)=\int_{-\infty}^{a} f(x)\,dx} = \underbrace{\int_{-\infty}^{0} \underbrace{f(x)}_{0}\cdot dx}_{0} + \int_{0}^{a} f(x)\cdot dx = 0 + \int_{0}^{a} x\cdot dx = \frac{x^2}{2}\Big|_{0}^{a} = \frac{a^2}{2}-\frac{0^2}{2} = \frac{a^2}{2}$$

III. $a \in \langle 1,2]$ (tj. $1 < a \le 2$)



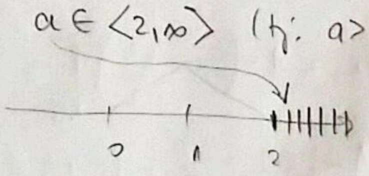
$$F(a)=\int_{-\infty}^{a} f(x)\,dx = \underbrace{\int_{-\infty}^{0} \underbrace{f(x)}_{0}dx}_{0} + \int_{0}^{1} f(x)\,dx + \int_{1}^{a} f(x)\,dx$$

$$F(a)=\int_{0}^{1} x\cdot dx + \int_{1}^{a}(2-x)\,dx = \frac{x^2}{2}\Big|_{0}^{1} + \left(2x-\frac{x^2}{2}\right)\Big|_{1}^{a} =$$

$$= \frac{1^2}{2}-\frac{0^2}{2} + 2a - \frac{a^2}{2} - \left(2\cdot 1-\frac{1^2}{2}\right) = \frac{1}{2} + 2a - \frac{a^2}{2} - 2 + \frac{1}{2} =$$

$$F(a) = -\frac{a^2}{2} + 2a - 1$$

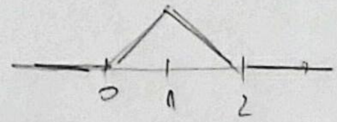
IV. $a \in \langle 2,\infty\rangle$ (tj. $a > 2$)



$$F(a)=\int_{-\infty}^{a} f(x)\,dx = \underbrace{\int_{-\infty}^{0} \underbrace{f(x)}_{0}dx}_{0} + \underbrace{\int_{0}^{1} f(x)\cdot dx + \int_{1}^{2} f(x)\,dx}_{\text{sva površina}=1} + \underbrace{\int_{2}^{a} \underbrace{f(x)}_{0}dx}_{0} = 1$$

$$F(a)=\begin{cases} 0, & a \le 0 \\ \frac{a^2}{2}, & 0 < a \le 1 \\ -\frac{a^2}{2}+2a-1, & 1 < a \le 2 \\ 1, & a > 2 \end{cases}$$

d) očekivanje:



$$\boxed{E(X) = \int_{-\infty}^{\infty} x \cdot f(x) \cdot dx} = 0 + \int_0^1 x f(x) dx + \int_1^2 x \cdot f(x) dx + 0$$

$$= \int_0^1 x \cdot x \cdot dx + \int_1^2 x \cdot (2-x) dx = \int_0^1 x^2 dx + \int_1^2 (2x - x^2) dx =$$

$$= \frac{x^3}{3}\Big|_0^1 + \left(2\frac{x^2}{\cancel{2}} - \frac{x^3}{3}\right)\Big|_1^2 = \frac{1^3}{3} - \frac{0^3}{3} + 2^2 - \frac{2^3}{3} - \left(1^2 - \frac{1^3}{3}\right) =$$

$$= \frac{1}{3} + 4 - \frac{8}{3} - 1 + \frac{1}{3} = 1$$

e)

$$\boxed{E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) dx} = 0 + \int_0^1 x^2 \cdot f(x) dx + \int_1^2 x^2 \cdot f(x) \cdot dx + 0 =$$

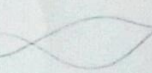
$$= \int_0^1 x^2 \cdot x \cdot dx + \int_1^2 x^2 \cdot (2-x) dx = \int_0^1 x^3 dx + \int_1^2 (2x^2 - x^3) dx$$

$$= \frac{x^4}{4}\Big|_0^1 + \left(2\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_1^2 = \frac{1^4}{4} - \frac{0^4}{4} + \frac{2 \cdot 2^3}{3} - \frac{2^4}{4} - \left(\frac{2 \cdot 1^3}{3} - \frac{1^4}{4}\right)$$

$$= \frac{1}{4} + \frac{16}{3} - 4 - \frac{2}{3} + \frac{1}{4} = \frac{7}{6}$$

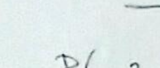
disperzija: $\boxed{D(X) = E(X^2) - [E(X)]^2} = \frac{7}{6} - 1^2 = \frac{7}{6} - 1 = \frac{1}{6}$

komentar: b) pomoću $F(a) = \begin{cases} 0 & , a \leq 0 \\ \frac{a^2}{2} & , 0 < a \leq 1 \\ -\frac{a^2}{2} + 2a - 1 & , 1 < a \leq 2 \\ 1 & , a > 2 \end{cases}$



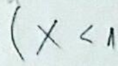
$P(a < X < b) = F(b) - F(a)$

$P(-3 \leq X < \frac{1}{3}) = F(\frac{1}{3}) - F(-3) = \frac{(\frac{1}{3})^2}{2} - 0 = \frac{1}{18}$



$P(X < a) = F(a)$

$P(X < 1{,}5) = F(1.5) = -\frac{(1.5)^2}{2} + 2 \cdot 1.5 - 1 = 0.875$



$P(X > a) = 1 - F(a)$

$P(X > -2) = 1 - F(-2) = 1 - 0 = 1$